

ONE PARTICLE BINDING OF MANY-PARTICLE SEMI-RELATIVISTIC PAULI-FIERZ MODEL

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ABSTRACT. It is shown that at least one particle is bound in the N -particle semi-relativistic Pauli-Fierz model with negative potential $V(\mathbf{x})$. It is assumed that the particles have no spin and obey the Bose or Boltzmann statistics, and the one particle Hamiltonian $\sqrt{-\Delta + M^2} - M + V(\mathbf{x})$ has a ground state with negative energy $-e_0 < 0$, where $M > 0$ denotes the mass of the particle. We show that the ground state energy of the total system $E^V(N)$ is less than $E^0(N) - e_0$.

1. INTRODUCTION

We consider a quantum system of N -charged relativistic particles interacting with the quantum electromagnetic field and the fixed nuclear potential $V(\mathbf{x}) \leq 0$. The Hamiltonian of this system is defined by

$$H^V := \sum_{i=1}^N T_{\mathbf{A}}(\mathbf{p}_i) + \sum_{i=1}^N V(\mathbf{x}_i) + H_f + \sum_{i < j} W(\mathbf{x}_i - \mathbf{x}_j), \quad (1.1)$$

where \mathbf{x}_i denotes the position of the i -th particle and $T_{\mathbf{A}}(\mathbf{p}_i)$ denotes the kinetic energy of the i -th particle which depends on the momentum $\mathbf{p}_i := -i\nabla_{\mathbf{x}_i}$ and the quantized electromagnetic potential $\mathbf{A}(\mathbf{x}_i)$. H_f denotes the free photon Hamiltonian and W is the interparticle potential energy. In this paper, the N -particles are treated as relativistic particles and we take the relativistic kinetic energy

$$T_{\mathbf{A}}(\mathbf{p}_i) = \sqrt{(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i))^2 + M^2} - M, \quad (1.2)$$

where q and M denote the charge and mass of the particle, respectively. The system described by the Hamiltonian (1.1) is called the semi-relativistic Pauli-Fierz model. We assume that the particles have no spin and obey the Bose-Einstein or Boltzmann statistics. We are interested in whether the nuclear potential V can bind the particle.

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Let $E^V(N)$ be the lowest energy of $H^V(N)$. Note that $E^0(N)$ is $E^V(N)$ with $V = 0$. In order to prove the existence of ground state for H^V , it is very important to show the inequality

$$E^V(N) < \min\{E^V(N - N') + E^0(N') \mid N' = 1, 2, \dots, N\}. \quad (1.3)$$

This inequality is called the binding condition. In this paper, we prove that

$$E^V(N) \leq E^0(N) - e_0, \quad (1.4)$$

where $-e_0$ is the ground state energy of the one particle Hamiltonian

$$h^V = \sqrt{-\Delta + M^2} - M + V(\mathbf{x}). \quad (1.5)$$

We assume that $-e_0 < 0$. Then (1.4) implies the strict inequality $E^V(N) < E^0(N)$, which is weaker case of (1.3). Physically, this inequality means that at least one particle is bound in the lowest energy state. When $N = 1$, the inequality (1.3) becomes $E^V(1) < E^0(1)$. The inequality $E^V(1) \leq E^0(1) - e_{\text{NR},0}$ was proved by [3] (see also [4]), where $-e_{\text{NR},0}$ is the ground state energy of the non-relativistic particle Hamiltonian $-(1/2M)\Delta + V$. The inequality $E^V \leq E^0 - e_0$ with the relativistic ground state energy $-e_0 := \inf \text{spec}(\sqrt{-\Delta + M^2} - M + V(\mathbf{x}))$ was shown in [1]. More better bound including the effect of the mass renormalization was given in [2]. The spectrum of the polaron of this relativistic model was studied by [8].

In the case where the particles have non-relativistic kinetic energy

$$T_{\mathbf{A}}(\mathbf{p}_i) = \frac{1}{2M}(\mathbf{p}_i - q\mathbf{A}(\mathbf{x}_i))^2, \quad (1.6)$$

the system is called the Pauli-Fierz model, and this model was widely studied. The most important result on the existence of ground state is the paper [7], where, for the non-relativistic case, it is proved that the inequality (1.3) implies the existence of ground state. The binding condition for an atomic Coulomb system was proved in the continuous paper [5]. The non-relativistic version of (1.4) was originally shown in [7]. Our result (1.4) can be considered as a relativistic improvement of [7, Theorem 3.1]. We have used the method they had developed in [7] with some modification. But, unfortunately, our method can not be applied for the Fermionic or spinor particles.

The difficulties to prove (1.4) come from the relativistic kinetic energy (1.2) which is clearly non-local. The key idea of our proof is to use the convexity of the kinetic energy $T_{\mathbf{A}}(\mathbf{p}_i)$ when estimating the energy expectation of a test function. The convexity of the kinetic energy follows from the property of the semi-group of the Hamiltonian which is positivity preserving([6]). The semi-group is, however, positivity preserving only for the case when the particles are spinless and obey the Bose-Einstein or Boltzmann statistics. But, it is remarkable that although the relativistic Schrödinger operator with a classical magnetic vector potential $\mathbf{A}(\mathbf{x})$ may not have the convexity, relativistic kinetic energy $T_{\mathbf{A}}(\mathbf{p}_i)$ is convex.

In Sect. 2, we give the rigorous definition of the system, and state the main result. In Sect. 3, we give the proof of the main theorem.

2. DEFINITION AND MAIN RESULT

The Hilbert space for the N -particle state is defined by

$$\mathcal{H}_{\text{part}} = L^2(\mathbb{R}^{3N}). \quad (2.1)$$

When the N particles obey the Bose-Einstein statistics, one needs to take $\otimes_{\text{sym}}^N L^2(\mathbb{R}^3)$ instead of (2.1) where \otimes_{sym} denotes the symmetric tensor product. Almost all the discussions in this paper are independent of the such choice of statistics. Hence we only consider the case of (2.1).

The position of the particles are denoted by $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N}$ with $\mathbf{x}_i = (x_i^1, x_i^2, x_i^3) \in \mathbb{R}^3$, $i = 1, \dots, N$. The Hilbert space for the photon field is the Fock space

$$\mathcal{H}_{\text{phot}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_{\text{sym}}^n L^2(\mathbb{R}^3 \times \{1, 2\}) \right], \quad (2.2)$$

with $\otimes_{\text{sym}}^0 L^2(\mathbb{R}^3 \times \{1, 2\}) =: \mathbb{C}$. The Hilbert space for the semi-relativistic Pauli-Fierz model is defined by

$$\mathcal{H} := \mathcal{H}_{\text{part}} \otimes \mathcal{H}_{\text{phot}}, \quad (2.3)$$

The smeared creation and annihilation operators in $\mathcal{H}_{\text{phot}}$ are denoted by $a(f)^*, a(f)$, $f \in L^2(\mathbb{R}^3 \times \{1, 2\})$, respectively. The vacuum vector is defined by $\Omega_{\text{phot}} := 1 \oplus 0 \oplus 0 \cdots \mathcal{H}_{\text{phot}}$. For a closed operator T on $L^2(\mathbb{R}^3 \times \{1, 2\})$, the second quantization of T is denoted by $d\Gamma(T) : \mathcal{H}_{\text{phot}} \rightarrow \mathcal{H}_{\text{phot}}$. Let $\omega : \mathbb{R}^3 \rightarrow [0, \infty)$ be a Borel measurable function such that $0 < \omega(\mathbf{k}) < \infty$. We also denote by the same symbol ω the multiplication operator by the function ω , which acts in $L^2(\mathbb{R}^3 \times \{1, 2\})$ as $(\omega f)(\mathbf{k}, \lambda) = \omega(\mathbf{k})f(\mathbf{k}, \lambda)$, $(\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \{1, 2\}$. The free Hamiltonian of the photon field is defined by

$$H_f := d\Gamma(\omega) \quad (2.4)$$

Let $\mathbf{e}^{(\lambda)} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\lambda = 1, 2$ be polarization vectors, which is defined by

$$\mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}^{(\mu)}(\mathbf{k}) = \delta_{\lambda, \mu}, \quad \mathbf{k} \cdot \mathbf{e}^{(\lambda)}(\mathbf{k}) = 0, \quad \mathbf{k} \in \mathbb{R}^3, \quad \lambda, \mu \in \{1, 2\}. \quad (2.5)$$

We write $\mathbf{e}^{(\lambda)}(\mathbf{k}) = (e_1^{(\lambda)}(\mathbf{k}), e_2^{(\lambda)}(\mathbf{k}), e_3^{(\lambda)}(\mathbf{k}))$ and suppose that each component $e_j^{(\lambda)}(\mathbf{k})$ is a Borel measurable function in \mathbf{k} . Let $\Lambda \in L^2(\mathbb{R}^3)$ be a function such that

$$\omega^{-1/2} \Lambda \in L^2(\mathbb{R}^3). \quad (2.6)$$

For $j = 1, 2, 3$, we set

$$g_j(\mathbf{k}, \lambda; \mathbf{x}) := \omega(\mathbf{k})^{-1/2} \Lambda(\mathbf{k}) e_j^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \{1, 2\}, \quad \mathbf{x} \in \mathbb{R}^3. \quad (2.7)$$

For each $\mathbf{x} \in \mathbb{R}^3$, $g_j(\mathbf{x}) = g_j(\cdot, \cdot; \mathbf{x})$ can be regarded as an element of $L^2(\mathbb{R}^3 \times \{1, 2\})$. Then, the quantized electromagnetic field at $\mathbf{x} \in \mathbb{R}^3$ is defined by

$$A_j(\mathbf{x}) := \frac{1}{\sqrt{2}} [a(g_j(\mathbf{x})) + a^*(g_j(\mathbf{x}))], \quad (2.8)$$

where \bar{T} denotes the closure of closable operator T . The quantized electromagnetic field $\mathbf{A}(\mathbf{x}) := (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x}))$ satisfies the Coulomb gauge condition:

$$\sum_{j=1}^3 \frac{\partial A_j(\mathbf{x})}{\partial x^j} = 0, \quad \mathbf{x} = (x^1, x^2, x^3). \quad (2.9)$$

The Hilbert space \mathcal{H} can be identified as

$$\mathcal{H} \cong \int_{\mathbb{R}^{3N}}^{\oplus} \mathcal{H}_{\text{phot}} d^{3N} \underline{X}, \quad \underline{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N}. \quad (2.10)$$

The quantized electromagnetic field on the total Hilbert space is defined by the fiber direct integral of $A_j(\mathbf{x})$:

$$A_j(\hat{\mathbf{x}}_i) := \int_{\mathbb{R}^{3N}}^{\oplus} A_j(\mathbf{x}_i) d\underline{X}. \quad (2.11)$$

Let C_c^∞ be the set of infinitely differentiable functions with compact support. Let

$$\mathcal{F}_{\text{fin}} := \mathcal{L}[\{a^*(f_1) \cdots a^*(f_n) \Omega_{\text{phot}}, \Omega_{\text{phot}} | f_j \in C_c^\infty(\mathbb{R}^3 \times \{1, 2\}), j = 1, \dots, n, n \in \mathbb{N}\}] \quad (2.12)$$

be a finite photon subspace spanned by $C_c^\infty(\mathbb{R}^3 \times \{1, 2\})$. The subspace

$$\mathcal{D} := (\hat{\otimes}_{\text{sym}}^N C_c^\infty(\mathbb{R}^3)) \hat{\otimes} \mathcal{F}_{\text{fin}} \quad (2.13)$$

is dense in \mathcal{H} , where $\hat{\otimes}$ denotes the algebraic tensor product. In what follows for notational convenience we omit the symbol \otimes in $L^2(\mathbb{R}^{3N}) \otimes \mathcal{H}_{\text{phot}}$. For two sets of operators $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$, we denote $\langle \mathbf{a}f, \mathbf{b}g \rangle$ by $\sum_{j=1}^3 \langle a_j f, b_j g \rangle$. We define the non-negative quadratic form on $\mathcal{D} \times \mathcal{D}$ by

$$K_{i,\mathbf{A}}(\Psi, \Phi) = \langle (\mathbf{p}_i - q\mathbf{A}(\hat{\mathbf{x}}_i))\Psi, (\mathbf{p}_i - q\mathbf{A}(\hat{\mathbf{x}}_i))\Phi \rangle + M^2 \langle \Psi, \Phi \rangle, \quad (2.14)$$

for $i = 1, \dots, N$, where $\mathbf{p}_i := -i\nabla_{\mathbf{x}_i} = -i(\partial_{x_i^1}, \partial_{x_i^2}, \partial_{x_i^3})$. Note that $K_{i,\mathbf{A}}$ is a positive closable form, and we denote its closure by $\bar{K}_{i,\mathbf{A}}$. Let $L_{i,\mathbf{A}}$ be the self-adjoint operator associated with $\bar{K}_{i,\mathbf{A}}$, i.e.,

$$\text{Dom}(L_{i,\mathbf{A}}^{1/2}) = Q(\bar{K}_{i,\mathbf{A}}), \quad (2.15)$$

$$\bar{K}_{i,\mathbf{A}}(\Psi, \Phi) = \left\langle L_{i,\mathbf{A}}^{1/2} \Psi, L_{i,\mathbf{A}}^{1/2} \Phi \right\rangle, \quad (2.16)$$

for all $\Psi, \Phi \in Q(\bar{K}_{i,\mathbf{A}})$. Since $\mathcal{D} \subset Q(L_{i,\mathbf{A}})$, we have $\mathcal{D} \subset \text{Dom}(L_{i,\mathbf{A}}^{1/2})$. We set

$$\sqrt{(\mathbf{p}_i - q\mathbf{A}(\hat{\mathbf{x}}_i))^2 + M^2} - M := T_{\mathbf{A}}(\mathbf{p}_i) := L_{i,\mathbf{A}}^{1/2} - M. \quad (2.17)$$

The Hamiltonian of the N -particle semi-relativistic Pauli-Fierz model is defined by

$$H^V := \sum_{i=1}^N \left(\sqrt{(\mathbf{p}_i - q\mathbf{A}(\hat{\mathbf{x}}_i))^2 + M^2} - M + V(\mathbf{x}_i) \right) + H_f + \sum_{1 \leq i < j \leq N} W(\mathbf{x}_i - \mathbf{x}_j), \quad (2.18)$$

where $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $W(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ are measurable functions. We set $H^0 := H^V|_{V=0}$. We introduce the following conditions:

- (H.1) $\omega^{3/2}\Lambda \in L^2(\mathbb{R}^3)$.
- (H.2) $V(\mathbf{x})$ and $W(\mathbf{x})$ are relatively compact with respect to the three dimensional relativistic Schrödinger operator $\sqrt{-\Delta_{\mathbf{x}} + 1} - 1$ and the relative bounds are strictly smaller than one.
- (H.3) The self-adjoint operator $h^V := \sqrt{-\Delta + M^2} - M + V(\mathbf{x})$ has a negative energy ground state $-e_0 < 0$.
- (H.4) $V(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^3$.

The essential self-adjointness was proved in [6, Corollary 7.60].

Proposition 2.1. *(Essential self-adjointness) Assume (H.1) and (H.2). Then, the Hamiltonians H^V, H^0 are essentially self-adjoint on \mathcal{D} .*

We denote the closure of H^V and H^0 by the same symbol. Let $E^0(N) = \inf \text{spec}(H^0)$ and $E^V(N) = \inf \text{spec}(H^V)$ are the ground state energies. The main result in this paper is the following:

Theorem 2.2. *Assume (H.1)–(H.4). Then, for all $q \in \mathbb{R}$ and $M \geq 0$, the inequality*

$$E^V(N) \leq E^0(N) - e_0 \quad (2.19)$$

holds.

3. PROOF OF THEOREM 2.2

We start from the the following basic fact.

Lemma 3.1. *Let (Q, Σ, μ) be a σ -finite measure space, and T be a positivity preserving bounded operator on $L^2(Q, d\mu)$. Then, for all non-negative $f, g \in L^2(Q, d\mu)$, the following holds*

$$(Tf)(q)^2 + (Tg)(q)^2 \leq [(T(f^2 + g^2)^{1/2})(q)]^2, \quad \mu\text{-a.e. } q \in Q. \quad (3.1)$$

Proof. First we assume that f, g are non-negative simple functions, i.e.,

$$f(q) = \sum_{i=1}^n \alpha_i \chi_{A_i}(q), \quad g(q) = \sum_{i=1}^n \beta_i \chi_{A_i}(q), \quad (3.2)$$

with $\alpha_i, \beta_i \geq 0$ and $A_i \in \Sigma$, where χ_{A_i} is the characteristic function of A_i . One can assume that $A_i \cap A_j = \emptyset$. Then, by noting that $T\chi_{A_i} \geq 0$, we have

$$(Tf)^2 + (Tg)^2 = \sum_i \sum_j (\alpha_i \alpha_j + \beta_i \beta_j) (T\chi_{A_i})(T\chi_{A_j}) \quad (3.3)$$

$$\leq \sum_i \sum_j (\alpha_i^2 + \beta_i^2)^{1/2} (\alpha_j^2 + \beta_j^2)^{1/2} (T\chi_{A_i})(T\chi_{A_j}) \quad (3.4)$$

$$= \left(T \sum_i (\alpha_i^2 + \beta_i^2)^{1/2} \chi_{A_i} \right)^2 \quad (3.5)$$

$$= \left(T \left(\sum_i \alpha_i^2 \chi_{A_i} + \sum_i \beta_i^2 \chi_{A_i} \right)^{1/2} \right)^2 \quad (3.6)$$

$$= (T(f^2 + g^2)^{1/2})^2 \quad (3.7)$$

For any $f, g \in L^2(Q, d\mu)$, there exist simple functions f_n, g_n such that $0 \leq f_n \leq f$, $0 \leq g_n \leq g$ and $f_n(q) \nearrow f(q)$, $g_n(q) \nearrow g(q)$, μ -a.e. q as $n \rightarrow \infty$. By (3.7), we have

$$(Tf_n)(q)^2 + (Tg_n)(q)^2 \leq [(T(f_n^2 + g_n^2)^{1/2})(q)]^2 \leq [(T(f^2 + g^2)^{1/2})(q)]^2 \quad (3.8)$$

for μ -a.e. $q \in Q$. Since T is bounded, we have $\|T(f - f_n)\| \rightarrow 0$ as $n \rightarrow \infty$. By taking the subsequence $\{n_j\}_j$, we have

$$\lim_{j \rightarrow \infty} (Tf_{n_j})(q)^2 + (Tg_{n_j})(q)^2 = (Tf)(q)^2 + (Tg)(q)^2 \leq [(T(f^2 + g^2)^{1/2})(q)]^2, \quad (3.9)$$

for μ -a.e. q . □

For a semi-bounded self-adjoint operator h , we denote the associated quadratic form by (f, hg) , $f, g \in Q(h)$. As a consequence of Lemma 3.1, we have the following fact:

Lemma 3.2. *Let h be a semi-bounded self-adjoint operator on a L^2 -space such that e^{-th} is positivity preserving for all $t > 0$. Then, for all $f \in \text{Dom}(h)$, $|f| \in Q(h)$ and*

$$(|f|, h|f|) \leq \langle f, hf \rangle. \quad (3.10)$$

In particular, for non-negative $f, g \in Q(h)$, $\sqrt{f^2 + g^2} \in Q(h)$ and

$$\left(\sqrt{f^2 + g^2}, h\sqrt{f^2 + g^2} \right) \leq (f, hf) + (g, hg) \quad (3.11)$$

holds.

Proof. Note that $u \in Q(h)$ if and only if $t^{-1} \langle u, (1 - e^{-th})u \rangle$ converges as $t \rightarrow 0$. Assume that $f \in \text{Dom}(h)$. Then

$$\langle f, hf \rangle = \lim_{t \rightarrow 0} t^{-1} (f, (1 - e^{-th})f) \geq \lim_{t \rightarrow 0} t^{-1} (|f|, (1 - e^{-th})|f|) = (|f|, h|f|) > -\infty, \quad (3.12)$$

which proves (3.10). Next we assume $f, g \in Q(h)$. By Lemma 3.1, we have

$$(f, hf) + (g, hg) = \lim_{t \rightarrow 0} t^{-1} [(f, (1 - e^{-th})f) + (g, (1 - e^{-th})g)] \quad (3.13)$$

$$\geq \lim_{t \rightarrow 0} t^{-1} (\sqrt{f^2 + g^2}, (1 - e^{-th})\sqrt{f^2 + g^2}) \quad (3.14)$$

$$= (\sqrt{f^2 + g^2}, h\sqrt{f^2 + g^2}) > -\infty, \quad (3.15)$$

which implies that $\sqrt{f^2 + g^2} \in Q(h)$ and (3.11) holds. \square

Proof of Theorem 2.2. By using the functional integration, it is proved that there exists a σ -finite measure space $(\mathcal{L}_E, \Sigma_E, \mu_E)$ and unitary operator $U : \mathcal{H}_{\text{phot}} \rightarrow L^2(\mathcal{L}_E, d\mu_E)$ such that $I \otimes U e^{-tH^0} I \otimes U^{-1}$ is positivity preserving (see [6, Corollary 7.64]). We set $\tilde{H}^0 = I \otimes U H^0 I \otimes U^{-1}$. For arbitrary fixed $\epsilon > 0$, we can choose normalized vectors $F \in \mathcal{D}$ and $\phi \in C_c^\infty(\mathbb{R}^3)$ such that

$$\langle F, H^0 F \rangle < E^0(N) + \epsilon \quad (3.16)$$

$$\langle \phi, h^V \phi \rangle < -e_0 + \epsilon \quad (3.17)$$

$$\phi(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^3. \quad (3.18)$$

For each $\mathbf{y} \in \mathbb{R}^3$, we define the translation operator

$$\mathcal{T}_{\mathbf{y}} := \exp \left(-i\mathbf{y} \cdot \sum_{i=1}^N \mathbf{p}_i \right) \otimes \exp(-i\mathbf{y} \cdot d\Gamma(\mathbf{k})). \quad (3.19)$$

One can show that $\mathcal{T}_{\mathbf{y}} \mathcal{D} = \mathcal{D}$ and H^0 is translation invariant, i.e., $\mathcal{T}_{\mathbf{y}}^{-1} H^0 \mathcal{T}_{\mathbf{y}} = H^0$. We set $\tilde{F} = (I \otimes U)F \in L^2(\mathcal{L}_E, d\mu_E)$. Our test function is

$$\Phi_{\mathbf{y}} = \left[\sum_{i=1}^N \phi(\hat{\mathbf{x}}_i)^2 \right]^{1/2} \mathcal{T}_{\mathbf{y}} I \otimes U^{-1} |\tilde{F}|, \quad (3.20)$$

where $\phi(\hat{\mathbf{x}}_i)$ denotes the multiplication operator by the function $\phi(\mathbf{x}_i)$. Note that

$$\int_{\mathbb{R}^3} d\mathbf{y} \|\Phi_{\mathbf{y}}\|^2 = \sum_{i=1}^N \|\phi(\mathbf{x}_i)\|_{L^2(\mathbb{R}^3)}^2 \cdot \|\tilde{F}\|^2 = N, \quad (3.21)$$

and

$$\int_{\mathbb{R}^3} d\mathbf{y} \left\langle \Phi_{\mathbf{y}}, \sum_{i=1}^N V(\mathbf{x}_i) \Phi_{\mathbf{y}} \right\rangle \quad (3.22)$$

$$= \sum_{i,j} \int_{\mathbb{R}^3} d\mathbf{y} \phi(\mathbf{x}_i + \mathbf{y})^2 V(\mathbf{x}_j + \mathbf{y}) \langle F(\mathbf{X}), F(\mathbf{X}) \rangle_{\mathcal{H}_{\text{phot}}} \quad (3.23)$$

$$\leq \sum_{i=1}^N \int_{\mathbb{R}^3} d\mathbf{y} \phi(\mathbf{x}_i + \mathbf{y})^2 V(\mathbf{x}_i + \mathbf{y}) \langle F(\mathbf{X}), F(\mathbf{X}) \rangle_{\mathcal{H}_{\text{phot}}} \quad (3.24)$$

$$= N \langle \phi, V\phi \rangle, \quad (3.25)$$

where we used the condition (H.4). By Lemma 3.2, we have $\Phi_{\mathbf{y}} \in Q(H^0)$ and

$$(\Phi_{\mathbf{y}}, H^0 \Phi_{\mathbf{y}}) = \left(\left[\sum_{i=1}^N \phi(\hat{\mathbf{x}}_i + \mathbf{y})^2 \right]^{1/2} |\tilde{F}|, \tilde{H}^0 \left[\sum_{i=1}^N \phi(\hat{\mathbf{x}}_i + \mathbf{y})^2 \right]^{1/2} |\tilde{F}| \right) \quad (3.26)$$

$$= \left(\left[\sum_{i=1}^N \phi(\hat{\mathbf{x}}_i + \mathbf{y})^2 |\tilde{F}|^2 \right]^{1/2}, \tilde{H}^0 \left[\sum_{i=1}^N \phi(\hat{\mathbf{x}}_i + \mathbf{y})^2 |\tilde{F}|^2 \right]^{1/2} \right) \quad (3.27)$$

$$\leq \sum_{i=1}^N \left(\phi(\hat{\mathbf{x}}_i + \mathbf{y}) |\tilde{F}|, \tilde{H}^0 \phi(\hat{\mathbf{x}}_i + \mathbf{y}) |\tilde{F}| \right) \quad (3.28)$$

$$\leq \sum_{i=1}^N \left(\phi(\hat{\mathbf{x}}_i + \mathbf{y}) \tilde{F}, \tilde{H}^0 \phi(\hat{\mathbf{x}}_i + \mathbf{y}) \tilde{F} \right) \quad (3.29)$$

$$= \sum_{i=1}^N \langle \phi(\hat{\mathbf{x}}_i + \mathbf{y}) F, H^0 \phi(\hat{\mathbf{x}}_i + \mathbf{y}) F \rangle \quad (3.30)$$

Lemma 3.3. For $i = 1, \dots, N$, we have

$$\int_{\mathbb{R}^3} d\mathbf{y} \langle \phi(\hat{\mathbf{x}}_i + \mathbf{y}) F, H^0 \phi(\hat{\mathbf{x}}_i + \mathbf{y}) F \rangle \quad (3.31)$$

$$\leq \langle F, H^0 F \rangle + \left\langle \phi, (\sqrt{-\Delta + M^2} - M) \phi \right\rangle_{L^2(\mathbb{R}^3)}. \quad (3.32)$$

Proof. The proof of the lemma is essentially same as the proof of [1, Corollary 3.3]. So we omit it. \square

By combining estimates (3.25), (3.30) and (3.32), we have

$$\int_{\mathbb{R}^3} d\mathbf{y} (\Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}}) \leq N \langle F, H^0 F \rangle + N \langle \phi, h^V \phi \rangle \quad (3.33)$$

$$\leq N(E^0(N) - e_0 + 2\epsilon), \quad (3.34)$$

which implies that there exist $\mathbf{y} \in \mathbb{R}^3$ such that $\|\Phi_{\mathbf{y}}\| \neq 0$ and

$$E^V(N)\|\Phi_{\mathbf{y}}\|^2 \leq (\Phi_{\mathbf{y}}, H^V \Phi_{\mathbf{y}}) < (E^0(N) - e_0 + 2\epsilon)\|\Phi_{\mathbf{y}}\|^2. \quad (3.35)$$

Since $\epsilon > 0$ is arbitrarily, inequality (2.19) holds. \square

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